# THE EFFECT OF IMMIGRATION ON WAITING TIME PROCESS FOR M/M/1 QUEUEING SYSTEM

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### *Abstract*

*In the present paper an attempt has been made to obtain the distribution of waiting time W(t) for the queue system M/M/1 under the assumption of the immigration effect. The distribution of the busy period and the joint distribution of the length of the busy period and the number of customers served during this busy period under this effect have been worked out.*

*Keywords: M/M/1 Queueing System, Immigration Effect, Waiting Time Process*

## 1. INTRODUCTION

The general problem of waiting time process under the queue system M/M/1 has been extensively studied. In the present work the effect of immigration on waiting time process W(t) for the queue system M/M/1 has been studied. Under this effect, the distribution of the busy period and the joint distribution of the length of the busy period and the number of customers served during this busy period have been worked out. Further, the distribution of waiting time W(t) has been investigated for the effect of immigration. The techniques of Laplace Stieltjes Transform and probability generating function have been used.

## 2. FORMULATION OF THE PROBLEM

Let us consider a queuing system in which customers arrive in a Poisson process at a rate $\lambda$ and are served by a single server on 'First-come, First-served' basis, the service time of a customer having negative exponential distribution $dB(t) = \mu e^{-\mu t}$ $(0 < t < \infty)$. The inter-arrival time is clearly $l/\lambda$ and the mean service time is $l/\mu$ so that the traffic intensity is $\rho = \lambda/\mu$. For the effect of immigration we take the assumption that in an infinitesimal time interval $\Delta t$ there is a chance $K\Delta t$ that a single member be added to the system by immigration from other sources. Then the new arrival rate and the service rate become $\lambda+K$ and $m$ instead of $\lambda$ and $\mu$ respectively.

## 3. SOLUTION OF THE PROBLEM

If A(t) denotes the number of arrivals during (0, t) then A(t) has the probability distribution.

$$\Pr[A(t) = n] = \frac{e^{-(1+K)t} \{1+K)t\}^n}{n!} \quad (n = 0, 1, 2, \ldots\ldots)$$

....(1)

Similarly, if $D(t)$ is the number of customers who complete their service and leave the system during $(0, t)$, then since the service time is distributed negative exponentially, $D(t)$ is also a Poisson process with probability distribution

$$\Pr[D(t) = n] = \frac{e^{-\mu t} (\mu t)^n}{n!} \quad (n = 0, 1, 2, \ldots\ldots)$$

....(2)

As we have $dB(t) = \mu\, e^{-\mu t}\, dt \quad (0 < t < \infty)$

....(3)

So, the n-fold convolution of the distribution $B(t)$ with itself is

$$B_n(t) = e^{-\mu t}\ \mu^n\ \frac{t^{n-1}}{(n-1)!} \qquad \text{for} \geq 1$$

....(4)

and

$$B_0(t) = 0 \qquad \text{if } t < 0$$
$$\qquad\quad = 1 \qquad \text{if } t \geq 0$$

Let $X(t)$ be the total service time of customers arriving in a time interval $(0, t)$; clearly $X(t) = t_1 + t_2 + \ldots + t_{A(t)}'$ where $A(t)$ is the number of arrivals during $(0, t)$, and has the Poisson distribution, whereas $t_1, t_2, \ldots\ldots,$ are mutually independent random variables with the distribution $dB(t)$. Thus the distribution of $X(t)$ has the Compound Poisson distribution as given by :

$$Q(0, t) = e^{-(\lambda+K)t}$$

$$Q(x, t) = \Pr[X(t) \leq x]$$

$$= \sum_{n=1}^{\infty}\ e^{-(\lambda+K)t}\ \frac{\{(\lambda+K)t\}^n}{n!}\ B_n(x)$$

and

$$d_x Q(x, t) \quad = \sum_{n=1}^{\infty}\ e^{-(\lambda+K)t}\ \frac{\{(\lambda+K)t\}^n}{n!}\ e^{-\mu x}\ \mu^n\ \frac{x^{n-1}}{(n-1)!}\,dx$$

$$= e^{-(\lambda+K)t - \mu x}\ \sum_{n=0}^{\infty}\ \{(\lambda+K)\mu\}^{n+1}\ \frac{t^{n+1}\, x^n}{n!\,(n+1)!}\,dx$$

or $\quad$ $d_x\,Q(x,\,t) \quad = e^{-(\lambda+K)t-\mu x}\,\sqrt{(\lambda+K)\,\mu\,t/x}\;I_1\,(2\sqrt{(\lambda+K)\,\mu tx})\,dx$

....(5)

where $I_j(x)$ is the modified Bessel function of the first kind of index j defined as

$$I_j(x) = \sum_{n=0}^{\infty}\frac{(x/2)^{2n+j}}{n!\,(n+j)!}$$

Thus X(t) has a discontinuity at x = 0 and has a continuous distribution in the range $0<x<\infty$, with the frequency function q(x, t), where

$$q(x,\,t) = e^{-(\lambda+K)t-\mu x}\,\sqrt{(\lambda+K)\,\mu\,t/x}\;I_1\,(2\sqrt{(\lambda+K)\,\mu tx})$$

....(6)

The Laplace Stieltjes Transform of x(t) is given by

$$E[e^{-\theta X(t)}] \quad = \int_0^{\infty}\,e^{-\theta x}\,d_x Q(x,\,t) = [e^{-\theta X(t)}] \quad = e^{-(\lambda+K)t+(\lambda+K)\,t\psi\,(\theta)}$$

....(7)

For the distribution of the busy period, initiated by a waiting time x, denoted by T(x), we have

$\quad$ T(x) = inf. $[t/x+X(t) - t \le 0]$

Also, let N(x) be the number of customers served during T(x), excluding those present initially. Let the joint distribution of T(x) and N(x) be denoted by

$\quad$ Gn(x, t) = Pr[T(x) < t; N(x) = n}

We have,

$\quad$ $dG_0(x,\,t) = \quad e^{-(l+K)t} \quad (t=x) \quad$ otherwise = 0

Since if no new customers arrive during (0, t), then the waiting time reduces to zero at time x. For $n \ge 1$, at least one new customer must arrive during (0, x), as otherwise the waiting time will reduce to zero at x< t; let the first new customer arrive at time $\tau$, where $\tau$ has the distribution $(\lambda+K)\,e^{-(\lambda+K)\tau}\,d\tau$ $(0<\tau<x)$. If $\upsilon$ m is the service time of this customer, then the waiting time $W(\tau + 0) = x - \tau + \upsilon$, where $\upsilon$ has the distribution $dB(\upsilon)$. During the remaining interval $(\tau, t)$, $\quad$ (n − 1) customers must be served, thus we have that

$\quad$ $dG_n(x,\,t) \quad = \int_{\tau=0}^{x}\,\int_{\upsilon=0}^{t-x}\,(\lambda+K)\,e^{-(\lambda+K)\tau}\,dG_{n-1}\,(x-\iota+\upsilon,\,t-\tau)d\tau\,dB(\upsilon) \quad (t \ge x)$

$\quad\quad\quad\quad = 0 \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (t < x)$

From which we have

$\quad$ $G_n(x,\,t) \quad = \int_{\tau=x}^{t}\,e^{-(\lambda+K)\tau}\,\frac{\{(\lambda+K)\tau\}^{n-1}}{n!}\,(\lambda+K)x.dB_n(\tau-x)\,(t \ge x)$

$$= 0 \qquad (t < x)$$

....(8)

From gives the distribution function of T(x) as

$$G(x, t) = \Pr[T(x) \le t] = \sum_{n=0}^{\infty} G_n(x, t)$$

$$\int_{\tau=x}^{t} \sum_{n=0}^{\infty} e^{-(\lambda+K)\tau} \frac{\{(\lambda+K)\tau\}^{n-1}}{n!} (\lambda + K)x.dB_n(\tau - x)$$

Thus, we have

$$dG(x, t) = e^{-(\lambda+K)x} \qquad (t = x)$$

$$= e^{-(\lambda+K)\,t - \mu x} \; x\sqrt{\frac{(\lambda+K)\mu}{t(t-x)}} \; I_1\,(2\sqrt{(\lambda+K)\,\mu t(t-x)}\, dt \; (t > x)$$

....(9)

It is obvious that in the range $x < t < \infty$, T(x) has a frequency function g(x, t) where

$$g(x, t) = \frac{x}{t} q(t - x, t)$$

....(10)

To obtain the Laplace Stieltjes Transform of T(x), we use the relation

$$T(x) = x + T[X(x)]$$

....(11)

Since, X(t) is a process with stationary independent increments, we have

$$E[e^{-\theta t(x)}] = [e^{-\theta T(1)}]x \; e^{-xn(\theta)} \qquad (\text{say})$$

on using equation (11), we have

$$e^{-xn(\theta)} = E[e^{-\theta T(x)}] = E[e^{-\theta x - n(\theta)X(x)}]$$

From equation (7), we have

$$e^{-xn(\theta)} = e^{-\theta x - (\lambda+K)x + (\lambda+K)x\,\psi(n)}$$

....(12)

Hence we see that n(q) satisfies the functional equation

$$n(\theta) = \theta + (\lambda+K) - (\lambda+K)\,\psi(n)$$

....(13)

As we know that the Laplace Transform of the service time distribution ($\mu\, e^{-\mu t}$; $\mu > 0$) is given by

$\psi(\theta) = \mu/\mu+\theta$, then the functional equation (12) may be written as–

$$n(\theta) = \theta + (\lambda+K) - (\lambda+K)\,\frac{\mu}{\mu+n} = n \qquad \text{(say)}$$

$$n(\mu+n) = \theta\,(\mu+n) + (\lambda+K)\,(\mu+n) - (\lambda+K)\mu$$

or, $\quad n^2 - n\{\theta + (\lambda+K) - \mu\} - \mu\theta = 0$

....(14)

So that the desired root is

$$n = n(\theta) = \frac{(\theta+\lambda+K-\mu) + \sqrt{(\theta+\lambda+K-\mu)^2 + 4\theta\mu}}{2}$$

....(15)

where the square root is taken so that its real part is positive, we have

$$n(0_+) = \frac{(\lambda+K-\mu) + |\lambda+K-\mu|}{2}$$

$$= \left\{ \begin{array}{ll} \lambda+K-\mu & \text{if } \rho > 1 \\ 0 & \text{if } \rho \le 1 \end{array} \right.$$

Thus the Laplace Stieltjes Transform of the distribution of T(x) is $\quad e^{-xn(\theta)}$ where $n(\theta)$ is given by equation (15) and

$$\Pr[T(x) < \infty] = \left\{ \begin{array}{ll} 1 & \text{if } \rho < 1 \\ e^{-(\lambda+K-\mu)x} & \text{if } \rho \le 1 \end{array} \right.$$

....(16)

Suppose that initially the system contains i ($\ge$ 1) customers, and let $T_i$ be the next subsequent epoch of time at which the server is free, $T_i$ is called the busy period initiated by i customers and $N(T_i)$ denotes the number of customers served during $T_i$.

The joint distribution of the length of the busy period $T_i$ and the number $N(T_i)$ of customers served during $T_i$ is given by :

$$\Pr[t < T_i < t + dt;\ N(T_i)=n] = e^{-(\lambda+K-\mu)t}\ \frac{i\lambda+K)^{n-i}\,\mu^n t^{2n-i-1}}{n(n-i)!(n-1)!}\ dt$$

....(17)

The distribution of $T_i$ found from this agrees with the frequency function $g_i(t)$ of the busy period $T_i$ which is given by :

$$g_i(t) = e^{-(\lambda+K-\mu)t}\ \rho^{-\left(\frac{i-1}{2}\right)}\ \left\{\frac{2i}{2\sqrt{(\lambda+K)\mu}}\right\}\ I_i(2\sqrt{(\lambda+K)\mu)\mu t})\,\mu$$

....(18)

$$(i \ge 1,\ t > 0)$$

and for the distribution $\{f_n\}$ of $N(T_1)$, the number of customers served during $T_1$, we have

$$f_n = \Pr[N(T_1) = n] = \int_{t=0}^{\infty} e^{-(\lambda + K)t} \frac{\{(\lambda + K)t\}^{n-1}}{n!} dB_n(t) \quad n \geq 1$$

$$= \int_{t=0}^{\infty} e^{-(\lambda + K)t} \frac{\{(\lambda + K)t\}^{n-1}}{n!} e^{-\mu t} \frac{\mu^n t^{n-1}}{(n-1)!} dt$$

$$= \frac{1}{(2n-1)} {}^{2n-1}C_{n-1} \frac{(\lambda + K)^{n-1}}{(\lambda + K + \mu)^{2n-1}} \quad n \geq 1$$

....(19)

The probability generating function (p.g.f.) of $N(T_1)$ is $\xi(z)$, where

$$\xi = \xi(z) = \frac{(\lambda + K + \mu) - \sqrt{(\lambda + K + \mu)^2 4(\lambda + K)\mu z}}{2(\lambda + K)}$$

....(20)

The probability that the system is empty at time t is given by

$$F(x; o, t) = \Pr[W(t) = 0/W(0) = x] \qquad (t \geq x \geq 0)$$

$$= e^{-(\lambda + K)t} + \sum_{n=1}^{\infty} e^{-(\lambda + K)t} \frac{\{(\lambda + K)t\}^{n-1}}{n!}$$

$$\int_0^{t-x} (\lambda + K)(t - \upsilon)\mu^n e^{-\mu\upsilon} \frac{\upsilon^{n-1}}{(n-1)!} d\upsilon$$

$$= e^{-(\lambda + K)t} + \int_0^t g(\upsilon, t)d\upsilon \qquad (t \geq x)$$

....(21)

Since, $dQ(\tau + x, t) = q(\tau + x, \tau) d\tau$, we can write the transition distribution function of the waiting time $W(t)$ as

$$F(x; y, t) = Q(t + y - x, t) - \int_0^{t-x} F(x; o, t - \tau) q(\tau + y, \tau) d\tau$$

....(22)

from which, for $y > 0$, we find that

$$\frac{\partial}{\partial x} F(x; y, t) = q(t + y - x, t) - \int_0^{t-x} F(x; o, t - \tau) q_1(\tau + y, \tau) d\tau,$$

where, for $x > 0$, on differentiating w.r.t. x to equation (6), we have

$$q_1(x, t) = \frac{\partial}{\partial x} q(x, t)$$

$$= -\mu q(x, t) - \frac{1}{2x} q(x, t) - e^{-(\lambda + K)t - \mu x} \left[ \frac{(\lambda + K)\mu t}{x} \right]$$

$$I_i (2\sqrt{(\lambda + K)\mu t x})$$

....(23)

Thus, W(t) has a discontinuity at the origin, whereas in the range 0<y<∞ it has a continuous distribution with the transition frequency function f(x; y, t) given by

$$f(x; y, t) = q(t+y-x, t) - \int_O^{t-x} F(x; o, t - \tau) \, q1 \, (\tau + y, \tau) d\tau$$

....(24)

## 4. CONCLUSION

The distribution of waiting time W(t) for the queue system M/M/1 under the assumption of the immigration effect has been obtained. When the effect of immigration is ignored in these results, one may get the same results as given by Prabhu (1965). The introduction of such an effect would change the results in terms of λ+K in place of λ. K–effect acts at an expected rate which is independent of the size the system has grown to.

## REFFERENCES

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